

Symmetric Graphs have symmetric Matchings

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Abstract

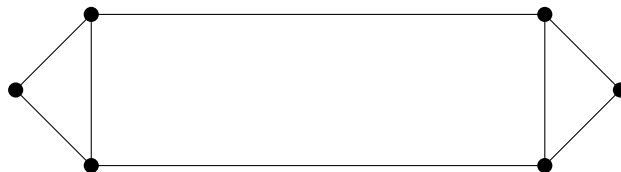
Assume that there is a free group action of automorphisms on a bipartite graph. If there is a perfect matching on the factor graph, then obviously there is a perfect matching on the graph. Surprisingly, the reversed is also true for amenable groups: if there is a perfect matching on the graph, there is also a perfect matching on the factor graph, i. e. a group invariant (“symmetric”) perfect matching on the graph.

1 Introduction

In 2007 we had an interesting discussion about a problem that somehow points to Erdős. Take \mathbb{Z}^2 in \mathbb{R}^2 and rotate that by 45° . Is there a bijection f between the points of the original lattice and the rotated lattice, such that the distance between x and $f(x)$ is bounded? We could confirm that (and that was already known), but is spurred us to find the smallest bound. Klaus Nagel found a nice construction of a bijection, and in a talk in July 2008 he proved that this one has the best bound, which is $\sqrt{5} \cdot \sin \frac{\pi}{8} \approx 0.8557$. Unfortunately, he never published that, and I made no notices.

Independent from that I tried to analyze the dependence of that bound from rotation angle. And, there are in fact two qualitatively different cases. If the rotation matrix is irrational, the “twin lattice” is aperiodic, and the bound doesn’t change if one of the lattices is translated. If the rotation matrix is rational, the “twin lattice” is periodic, and the bound depends on the rotation center. Nevertheless there is some continuous dependence of that bound from the rotation angle, and it is a nice fractal function. Since I cannot prove all of that it is still unpublished, but I will write a survey paper soon.

In doing numerical experiments on that problem, I faced the problem of finding a matching in an infinite but periodic bipartite graph. So I only considered periodic matchings which could be done by the well-known Hopcroft-Karp algorithm. But does that always work? Consider the following graph with a rotation symmetry:



It has two different perfect matchings, but none of them has the rotation symmetry.

So I was thinking about that in 2009 while skiing at the Kvitfjell in Norway, when I found out, that it works on bipartite graphs using the Theorem of Hall, because the “boundary” of a suitable finite subset of \mathbb{Z}^2 is “small” relative to the “interior”. So I could prove a general result for groups with subexponential growth. I suspected that it has something to do with amenability and unsuccessful tried to find a counterexample for the free group of two generators. So it remained unpublished.

In April 2016 I wanted to present that proof in a lecture, and suddenly I found the slight modification of the proof for showing it for all amenable groups, and the paradoxical decomposition gave me the right hint for counterexamples for all non-amenable groups. You will find that proof and the counterexamples in the last section. In the sections in between I collect the well-known facts about the Theorem of Hall (section 2), define symmetry of a bipartite graph and the factor graph (section 3), and cite equivalent characterizations of amenability and a slight modification of one of them (section 4).

I want to thank Rainer Rosenthal, who still patiently waits for Klaus Nagel’s paper and my survey paper.

2 Matchings

A *bipartite graph* (A, B, E) is a pair of sets A and B with a subset $E \subseteq A \times B$. The elements of A and B are *vertices*, the elements of E are *edges*. For each subset $X \subseteq A$ we define $E(X) = \{y \in B \mid (x, y) \in E\}$ and for each $Y \subseteq B$ we define $E(Y) = \{x \in A \mid (x, y) \in E\}$. Such a bipartite graph is called *locally finite*, if for any finite $X \subseteq A$ also $E(X)$ is finite, and for any finite $Y \subseteq B$ also $E(Y)$ is finite. (Equivalently, for any $x \in A$ and $y \in B$ the sets $E(\{x\})$ and $E(\{y\})$ are finite.)

A bipartite graph (A, B, E) fulfills the *left Hall condition*, if $|X| \leq |E(X)|$ for any finite $X \subseteq A$, and the *right Hall condition*, if $|Y| \leq |E(Y)|$ for any finite $Y \subseteq B$. It fulfills the *Hall condition*, if the left and right Hall condition are fulfilled.

A *matching* M of a bipartite graph is a subset $M \subseteq E$, such that for all $(x_1, y_1), (x_2, y_2) \in M$ there is $(x_1, y_1) = (x_2, y_2)$ whenever $x_1 = x_2$ or $y_1 = y_2$. Define $A(M) = \{x \in A \mid \exists y \in B : (x, y) \in M\}$ and $B(M) = \{y \in B \mid \exists x \in A : (x, y) \in M\}$.

$(x, y) \in M\}$. A matching M is called *perfect*, if $A(M) = A$ and $B(M) = B$.

Theorem 2.1 (Hall). *For a locally finite bipartite graph there are equivalent:*

1. *The graph fulfills the Hall condition.*
2. *The graph has a perfect matching.*

3 Symmetry

Definition 3.1. Let G be a group and (A, B, E) a bipartite graph. Then the graph is called *G-symmetric*, if

- G acts free on A ,
- G acts free on B ,
- $(x, y) \in E \iff (gx, gy) \in E$ for all $x \in A, y \in B$ and $g \in G$.

It is called *proper G-symmetric*, if over-more

- $(x, y) \in E$ and $(gx, y) \in E$ implies g is the identity, or equivalently
- $(x, y) \in E$ and $(x, gy) \in E$ implies g is the identity.

Definition 3.2. A matching M on a G -symmetric bipartite graph is called *G-symmetric*, if $(x, y) \in M$ implies $(gx, gy) \in M$ for any $g \in G$.

A G -symmetric matching can also be described by a matching on some factor graph:

Definition 3.3. Let (A, B, E) a G -symmetric bipartite graph. Then the *factor graph* $(\tilde{A}, \tilde{B}, \tilde{E})$ is defined by:

- $\tilde{A} = A/G = \{Gx \mid x \in A\}$,
- $\tilde{B} = B/G = \{Gy \mid y \in B\}$,
- $\tilde{E} = \{(Gx, Gy) \mid (x, y) \in E\}$.

Lemma 3.4. *If (A, B, E) is locally finite, then also $(\tilde{A}, \tilde{B}, \tilde{E})$ is locally finite.*

Proof. Let (A, B, E) be locally finite. Then

$$\begin{aligned} \tilde{E}(\{Gx\}) &= \{Gy \mid (Gx, Gy) \in \tilde{E}\} = \{Gy \mid (gx, y) \in E \text{ and } g \in G\} \\ &= \{Gy \mid (x, g^{-1}y) \in E \text{ and } g \in G\} = \{Gy' \mid (x, y') \in E\}, \end{aligned}$$

hence finite. The same holds for $\tilde{E}(\{Gy\})$. □

Proposition 3.5. *The map $M \mapsto \tilde{M} = \{(Gx, Gy) \mid (x, y) \in M\}$ is a surjection between the G -symmetric matchings M of (A, B, E) and the matchings \tilde{M} of $(\tilde{A}, \tilde{B}, \tilde{E})$. Is (A, B, E) even proper G -symmetric this map is a bijection. Overmore, M is perfect if and only if \tilde{M} is perfect.*

4 Amenable groups

There are some equivalent definitions of amenable groups using means on groups or invariant probability measures. Since that properties doesn't matter here, we just choose that one:

Definition 4.1. A group G is called *amenable*, if there is an invariant finitely additive probability measure on G .

The property “amenable” is equivalent to some other nice properties of the group. In particular, we need the Følner condition and the paradoxical decomposition.

Definition 4.2. For a group G note by \mathcal{F} the family of all non-empty finite subsets.

A group G satisfies the *Følner condition*, if for any $\varepsilon > 0$ and any $U \in \mathcal{F}$ there is a $F \in \mathcal{F}$ with $|F \setminus Fg| \leq \varepsilon \cdot |F|$ for any $g \in U$.

A group G has a *paradoxical decomposition*, if there is a $F \in \mathcal{F}$ and a decomposition

$$G = \bigsqcup_{g \in F} A_g = \bigsqcup_{g \in F} B_g = \bigsqcup_{g \in F} A_g g \sqcup \bigsqcup_{g \in F} B_g g.$$

Theorem 4.3 (Følner, Tarski). *For a group G are equivalent:*

1. G is amenable.
2. G satisfies the Følner condition.
3. G has no paradoxical decomposition.

Both of that properties are interesting here, but we need a slight modification of the Følner condition:

Proposition 4.4. *A group G satisfies the Følner condition if and only if*

$$\inf_{F \in \mathcal{F}} \frac{|FU|}{|F|} = 1$$

for any $U \in \mathcal{F}$, where \mathcal{F} is the family of all non-empty finite subsets of G .

Proof. Note that $|F| \leq |FU|$ for all $F, U \in \mathcal{F}$, so $\inf_{F \in \mathcal{F}} \frac{|FU|}{|F|} \geq 1$.

“ \Rightarrow ”: Let $U \in \mathcal{F}$ and $\varepsilon > 0$. By the Følner condition there is a set $F \in \mathcal{F}$ such that

$$|F \setminus Fg| \leq \frac{\varepsilon}{|U|} \cdot |F|$$

for any $g \in U$. Since $|F| = |Fg|$ we have $|F \setminus Fg| = |Fg \setminus F|$, hence

$$|FU \setminus F| = \left| \left(\bigcup_{g \in U} Fg \right) \setminus F \right| = \left| \bigcup_{g \in U} (Fg \setminus F) \right| \leq \varepsilon \cdot |F|.$$

Now

$$|FU| = |(FU \setminus F) \cup (FU \cap F)| = |FU \setminus F| + |FU \cap F| \leq \varepsilon \cdot |F| + |F|$$

implies the assertion.

“ \Leftarrow ”: Let $U \in \mathcal{F}$ and $\varepsilon > 0$. Then there is a set $F \in \mathcal{F}$ such that

$$|F(U \cup \{e\})|/|F| < 1 + \varepsilon.$$

For any $g \in U$ we have

$$|F(U \cup \{e\})| = |FU \cup F| = |(FU \setminus F) \cup F| = |FU \setminus F| + |F| \geq |Fg \setminus F| + |F|.$$

This implies $|F \setminus Fg| = |Fg \setminus F| < \varepsilon \cdot |F|$. \square

Example 4.5. Finite groups are amenable.

Solvable groups are amenable, in particular abelian and nilpotent groups are amenable.

Any subgroup of an amenable group is amenable.

If U is an amenable normal subgroup of G , and G/U is amenable too, then G is amenable.

Any group with subexponential growth is amenable.

Example 4.6. Any free group on at least two generators is not amenable.

Any group that has a non-amenable subgroup is not amenable, e.g. $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 2$, since it has a subgroup isomorphic to the free group on two generators.

5 Symmetric graphs have symmetric matchings

Now we can formulate and proof the main result:

Theorem 5.1. *Let (A, B, E) a locally finite G -symmetric bipartite graph, where G is amenable. Then the following properties are equivalent.*

1. (A, B, E) has a perfect matching.
2. (A, B, E) has a perfect G -symmetric matching.

Since the reverse direction is obvious, there is only one direction to show.

Proof. Let $\tilde{X} \subseteq \tilde{A}$ be some finite subset in the factor graph. Set $\tilde{Y} = \tilde{E}(\tilde{X})$. Then there are representing sets $X \subseteq A$ and $Y \subseteq B$, i.e.

$$\tilde{X} = \{Gx \mid x \in X\}, \quad \tilde{Y} = \{Gy \mid y \in Y\}, \quad |X| = |\tilde{X}|, \quad |Y| = |\tilde{Y}|.$$

Now for any pair $(x, y) \in E(X)$ there is some $g \in G$ with $gy \in Y$, so there is a $U \in \mathcal{F}$ with $E(X) \subseteq UY$. Since E is G -invariant, we conclude $E(FX) \subseteq FUY$ for any $F \in \mathcal{F}$.

By the left Hall condition, the free action of G , and the modified Følner condition this implies

$$|F| \cdot |X| \leq |FUY| \cdot |Y| \implies |X| \leq \frac{|FUY|}{|F|} \cdot |Y| \implies |X| \leq |Y| \implies |\tilde{X}| \leq |\tilde{Y}|$$

which proves the left Hall condition in the factor graph. The right Hall condition is shown the same way, so the assertion is proved. \square

Proposition 5.2. *If G is not amenable, then there is a locally finite proper G -symmetric bipartite graph (A, B, E) with a perfect matching, that admits no G -symmetric perfect matching.*

Proof. By the Tarski Theorem there is a paradoxical decomposition of G . This can be used to construct such a G -symmetric bipartite graph: Set $A = G$, $B = G \times \{1, 2\}$, $E = \{(xg, (x, i)) \mid x \in G, g \in F, i \in \{1, 2\}\}$, and the group action is the group composition. Then there is a perfect matching

$$M = \{(xg, (x, 1)) \mid g \in F, x \in A_g\} \cup \{(xg, (x, 2)) \mid g \in F, x \in B_g\}.$$

But the factor graph $\tilde{A} = \{G\}$, $\tilde{B} = \{G\} \times \{1, 2\}$, $\tilde{E} = \tilde{A} \times \tilde{B}$ has obviously no perfect matching.

It's easy to see that this example is not proper G -symmetric, so we have to modify it slightly. Each vertex has to be replaced by $|F|$ copies of itself, and the edges have to be twisted a little bit.

Let $\varphi : F \times F \rightarrow F$ be bijective if any of the arguments is fixed. One can get such a function for instance by the group composition in a cyclic group of order $|F|$, and map the elements bijectively to F . Now we set $A = G \times F$, $B = G \times F \times \{1, 2\}$, and

$$E = \{((xg, \varphi(g, h)), (x, h, i)) \mid g, h \in F, x \in G, i \in \{1, 2\}\}.$$

G acts again by left multiplication on the G -component of A and B . This G -symmetry is proper, because

$$((z x g, \varphi(g, h)), (x, h, i)) = ((x' g', \varphi(g', h')), (x', h', i'))$$

for some $z \in G$ implies $x = x'$, $h = h'$, $i = i'$, and by injectivity of $g \mapsto \varphi(g, h)$ also $g = g'$, hence $z = e$.

A perfect matching is given by

$$M = \{((xg, \varphi(g, h)), (x, h, 1)) \mid g, h \in F, x \in A_g\} \cup \\ \{((xg, \varphi(g, h)), (x, h, 2)) \mid g, h \in F, x \in B_g\} \subseteq E.$$

Since $G = \bigsqcup_{g \in F} A_g = \bigsqcup_{g \in F} B_g$ any vertex of B , and since $G = \bigsqcup_{g \in F} A_g g \sqcup \bigsqcup_{g \in F} B_g g$ and $h \mapsto \varphi(g, h)$ is bijective, any vertex of A belongs to exactly one edge in M . Hence, (A, B, E) admits a perfect matching.

Now take a look at the factor graph. We have $\tilde{A} = \{G\} \times F$, $\tilde{B} = \{G\} \times F \times \{1, 2\}$,

$$\tilde{E} = \{((G, \varphi(g, h)), (G, h, i)) \mid g, h \in F, i \in \{1, 2\}\},$$

which is the complete bipartite graph with $|F|$ and $2|F|$ vertices. That one has obviously no perfect matching. \square

References

- [1] T. Ceccherini-Silberstein, M. Coornaert: *Cellular Automata and Groups*. Springer 2010.